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On Nonuniqueness in Rational L_p-Approximation

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The method described by I. Diener [3] is applied to rational functions rather than to families with one nonlinear parameter. Given an $(m+2)$-dimensional subspace of L_p, a function with two or more nearest points in R_n^m is obtained by the Borsuk antipodality theorem, if $n>0$ and some exceptional cases are excluded. Moreover, a symmetry argument leads to functions with at least four global solutions. © 1987 Academic Press, Inc.

In the preceding paper, Diener obtained a nonuniqueness result (Theorem 3.3) by using the following topological argument. Assume that for each f in some subset E_1 of $L_2(X)$ there is a unique nearest point in some approximatively compact set $M \subset L_2(X)$. Then the metric projection $P: E_1 \to M$ is continuous, and a fixed point theorem yields a degenerate best approximation for some $f \in E_1$. Since this can be ruled out, a contradiction to the uniqueness hypothesis is obtained.

In this note, we will establish a general nonuniqueness result for rational approximation by using the Borsuk antipodality theorem. As in [3], R_n^m denotes the family of rational functions whose numerators and denominators are polynomials of degree $\leqslant m$ and $\leqslant n$, respectively.

THEOREM 1. *Let $1<r<\infty$, $m \geqslant 0$, $n \geqslant 1$, and $[\alpha, \beta]$ be a nondegenerate real interval. Each $(m+2)$-dimensional subspace E_1 of $L_r[\alpha, \beta]$ such that $E_1 \cap R_n^m = \{0\}$, contains a function with at least two best approximations from R_n^m.*

Our proof of the theorem will make use of the

BORSUK ANTIPODALITY THEOREM. *Let Ω be a bounded, open, symmetric neighborhood of 0 in $\mathbb{R}^{N+1}$, and $T: \partial\Omega \to \mathbb{R}^N$ be an odd, continuous mapping. Then there exists $x^* \in \partial\Omega$ for which $T(x^*)=0$.*

A mapping T is said to be odd if $T(-x) = -T(x)$.

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Proof of Theorem 1. The set $S^{m+1} := \{f \in E_1 : \|f\| = 1\}$ is homeomorphic to an $(m+1)$-dimensional sphere. Suppose that to each $f \in E_1$ there is a unique L_r-nearest point in R_n^m. It is well known [1, 2] that a best approximation is not contained in R_{n-1}^{m-1} and that R_n^m is approximatively compact. By standard arguments [3], the metric projection $P: S^{m+1} \to R_n^m$ is continuous. Let the denominators be normalized by $\max_{x \in [\alpha,\beta]} \{q(x)\} = 1$. Then the mapping $\phi: R_n^m \backslash R_{n-1}^{m-1} \to \mathbb{R}^{m+1}$ which sends $u = p/q$ with $p = \sum_{k=0}^{m} a_k x^k$ to the vector of coefficients $(a_0, a_1, ..., a_m)$, is continuous. Obviously, $P(-f) = -P(f)$. Therefore, $\phi \circ P$ is an odd mapping. By the Borsuk antipodality theorem, there exists an $f_0 \in S^{m+1}$ such that $\phi \circ P(f_0) = 0$. However, this equation implies that $Pf_0 \in R_{n-1}^{m-1}$, and we have a contradiction. Thus, P cannot be continuous and not each $f \in E_1$ has a unique nearest point in R_n^m. ▮

Since the nonlinear set R_n^m contains $(m+1)$-dimensional linear subsets and the best L_r-approximation in linear subsets is unique, it is natural that one gains nonuniqueness only modulo $(m+1)$-dimensional spaces. It would be interesting to have some knowledge about the structure of the set with those elements in $(m+3)$-dimensional subspaces, which have more than one nearest point.

THEOREM 2. *Let* $1 < r < \infty$ *and* $m \geq 0$. *Let* E_1 *be an* $(m+2)$*-dimensional subspace of* $L_r[-1, +1]$ *such that* $E_1 \cap R_1^m = \{0\}$ *and*

$$f(-x) = (-1)^{m+1} f(x) \qquad \text{for each } f \in E_1.$$

Then E_1 *contains a function with at least four best approximations from* R_1^m.

Proof. Let

$$M = \{u = p/q \in R_1^m : q(x) = b_0 + b_1 x, b_1 \geq 0\}.$$

It follows from the well-known symmetry argument that each best approximation to $f \in E_1$ from M is also a best approximation from R_1^m. Moreover, no best approximation is degenerate. Hence, M is approximatively compact.

The same arguments as in the preceding proof imply that there exists an $f \in E_1$ with two best approximations u_1 and u_2 from M. By setting $u_3(x) = (-1)^{m+1} u_1(-x)$ and $u_4(x) = (-1)^{m+1} u_2(-x)$ we obtain two more best approximations from R_1^m. ▮

Similarly, odd functions with four best approximations from R_2^0 may be constructed.

Note added in proof. With the same arguments as in Theorem 1 we conclude that each $(m+2)$-dimensional subspace of $C[\alpha, \beta]$ contains an element for which the best uniform approximation from R_n^m is degenerate.

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